MATHEMATICS AND MODELLING REFRESHER COURSE PRACTICE PROBLEMS

Probability and Statistics

Probability

3-1 What is the probability of getting three sixes in a throw of three dice? ANS $\frac{1}{216}$

3-2 A wooden cube with painted faces is sawed up into 1000 little cubes, all of the same size. The little cubes are then mixed up, and one is chosen at random. What is the probability of its having just 2 painted faces? ANS 0.096

3-3 Ten books are placed in random order on a bookshelf. Find the probability of three given books being side by side. ANS 0.0666

3-4 In playing Lotto 6/49, you can select 6 numbers from 49 numbers $(1, 2, \ldots, 49)$. On the day of drawing, six regular numbers and one bonus numbers are drawn. You win if the numbers you selected match at least three of the regular numbers drawn.

(1) What is the probability of winning the jackpot (matching all 6 numbers) of Lotto 6/49?

(2) What is the probability of winning the second prize (matching 5 regular numbers plus the bonus number) of Lotto 6/49?

(3) What is the probability of winning the third prize (matching 5 numbers) of Lotto 6/49?

(4) What is the probability of winning the fourth prize (matching 4 numbers) of Lotto 6/49?

(5) What is the probability of winning the fifth prize (matching 3 numbers) of Lotto 6/49?

ANS (1) $\frac{1}{13{,}983{,}816}$ (2) $\frac{6}{13{,}983{,}816}$ (3) $\frac{252}{13{,}983{,}816}$ (4) $\frac{13{,}545}{13{,}983{,}816}$ (5) $\frac{246{,}820}{13{,}983{,}816}$

3-5 A farmer came to the Water Resources Laboratory of the University of Waterloo. The farmer brought a carved whalebone with which he claimed that he could locate hidden sources of water. The following experiment was conducted to test the farmer's claim. He was taken into a room in which there were 10 covered cans. He was told that 5 of the 10 cans contained water and 5 were empty. The farmer's task was to divide the cans into two equal groups, 1 group containing all the cans with water, the other containing those without water. What is the probability that the farmer correctly put at least 3 cans into the water group just by chance? ANS 0.5

3-6 Fifteen people independently choose a number between 1 and 100. If each number has an equal chance of being chosen by any person, what is the probability that no two or more people will choose the same number. ANS 0.3313

3-7 Let A be the event that a card picked at random from a full deck is a spade, and B the event that it is a queen. Are A and B independent events?

3-8 The probability of a man aged 60 dying within one year is 0.025, and the probability of a woman aged 55 dying within one year is 0.011. If a man and his wife are 60 and 55 respectively, what is the probability of their both living a year? ANS 0.9643

3-9 A power plant has two generating units, numbered 1 and 2. Because of maintenance and occasional machine malfunctions, the probabilities that, in a given week, units No. 1 and 2 will be out of service (these two events are denoted by E_1 and E_2) are 0.01 and 0.02, respectively. During a summer week there is a probability of 0.10 that the weather will be extremely hot (say average temperature > 35°C; this event is denoted by H) so that demand for power for air-conditioning will increase considerably. Assume that H, E_1 and E_2 are statistically independent. The performance of the power plant in terms of its potential ability to meet the demand in a given week can be classified as

- satisfactory S — if both units are functioning and the average temperature is below 35°C
- poor P — if one (only one) of the units is out of service and the average temperature is above 35°C
- otherwise M — otherwise (not S or P)

(1) Define the events S, P, and M in terms of H, E_1 and E_2.

(2) What is the probability that exactly one unit will be out of service in any given week?

(3) Find $P(S), P(P)$, and $P(M)$.

ANS (2) 0.0296 (3) $P(S) = 0.87318$, $P(P) = 0.00296$, $P(M) = 0.12386$

3-10 In a building construction project, the completion of the building requires the successive completion of a series of activities. Define

- $E =$ excavation completed on time; and $P(E) = 0.8$
- $F =$ foundation completed on time; and $P(F) = 0.7$
- $S =$ superstructure completed on time; and $P(S) = 0.9$

Assume statistical independence among these events.

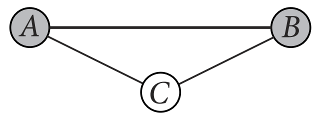
(1) Define the event $C = \{\text{project completed on time}\}$ in terms of E, F, and S. Compute the probability $P(C)$.

(2) Define the event $G = \{\text{excavation will be on time and at least one of the other two operations will not be on time}\}$ in terms of E, F, and S and their complements. Calculate $P(G)$.

(3) Define the event $H = \{\text{only one of the three operations will be on time}\}$ in terms of E, F, and S and their complements. Determine the probability $P(H)$.

ANS (1) 0.504 (2) 0.296 (3) 0.092

3-11 The highway system between cities A and B is shown in the following figure. Travel between A and B during the winter months is not always possible because some parts of the highway may not be open to traffic due to extreme weather condition. Let E_1, E_2, and E_3 denote the events that highway AB, AC, and CB are open, respectively.



On any given day, assume

$$P(E_1) = \frac{1}{5}, \quad P(E_2) = \frac{3}{4}, \quad P(E_3) = \frac{2}{3}, \quad P(E_3 \mid E_2) = \frac{4}{5}, \quad P(E_1 \mid E_2E_3) = \frac{1}{2}$$

(1) What is the probability that a traveller will be able to make a trip from A to B if he has to pass through city C?

(2) What is the probability that he will be able to get to city B?

(3) Which route should he try first in order to maximize his chance of getting to B?

ANS (1) 0.6 (2) 0.7

3-12 The water supply system for a city consists of a storage tank and a pipe line supplying water from a reservoir some distance away. The amount of water available from the reservoir is variable depending on the precipitation in the watershed (among other things). Consequently, the amount of water stored in the tank would be also variable. The consumption of water also fluctuates considerably. Let

- A = available water supply from the reservoir is low, $P(A) = 0.20$
- B = water stored in the tank is low, $P(B) = 0.15$
- C = level of consumption is low, $P(C) = 0.50$

The reservoir supply is regulated to a certain extent to meet the demand, so that

$$P(\bar{A} \mid \bar{C}) = P(\text{reservoir supply is high} \mid \text{consumption is high}) = 0.75$$

Also, $P(B \mid A) = 0.5$, whereas the amount of water stored is independent of the demand. Suppose that a water shortage will occur when there is high demand (or consumption) for water and either the reservoir supply is low or the stored water is low. What is then the probability of a water shortage? Assume that $P(AB \mid \bar{C}) = 0.5\,P(AB)$. ANS 0.175

3-13 A gravity retaining wall may fail (F) either by sliding (A) or overturning (B) or both. Assume:

- Probability of failure by sliding is twice as likely as that by overturning, i.e., $P(A) = 2P(B)$.
- Probability that the wall also fails by sliding, given that it has failed by overturning, $P(A \mid B) = 0.8$.
- Probability of failure of wall $P(F) = 10^{-3}$.

(1) Determine the probability that sliding will occur.

(2) If the wall fails, what is the probability that only sliding has occurred?

ANS (1) 0.00091 (2) 0.546

3-14 A town is protected from floods by a reservoir dam that is designed for a 50-year flood; that is the probability that the reservoir will overflow in a year is 1/50 or 0.02. The town and the reservoir are located in an active seismic region; annually, the probability of occurrence of a destructive earthquake is 5%. During such an earthquake, it is 20% probable that the dam will be damaged, thus causing the reservoir water to flood the town. Assume that the occurrences of natural floods and earthquakes are statistically independent.

(1) What is the probability of an earthquake-induced flood in a year?

(2) What is the probability that the town is free from flooding in any one year?

(3) If the occurrence of an earthquake is assumed in a given year, what is the probability that the town will be flooded that year?

ANS (1) 0.01 (2) 0.9702 (3) 0.216

3-15 One urn contains 40 white balls and 60 black balls, while another urn contains 10 white balls and 50 black balls. An urn is selected at random, and then a ball is drawn (at random) from the urn. The ball turns out to be white, and is then put back into the urn. What is the probability that another ball drawn from the same urn will be black? ANS 0.6686

3-16 There are three modes of transporting material from Ontario to Florida, namely, by land, sea, or air. Also land transportation may be by rail or highway. About half of the materials are transported by land, 30% by sea, and the rest by air. Also, 40% of all land transportation is by highway and the rest by rail shipments. The percentages of damaged cargo are, respectively, 10% by highway, 5% by rail, 6% by sea, and 2% by air.

(1) What percentage of all cargoes may be expected to be damaged?

(2) If a damaged cargo is received, what is the probability that it was shipped by land? By sea? By air?

ANS (1) 0.057 (2) $P(L \mid D) = 0.614$, $P(S \mid D) = 0.316$, $P(A \mid D) = 0.070$

Discrete Probability Distributions

3-17 The sewers in a city are designed for a rainfall having a return period of 10 years.

(1) What is the probability that the sewers will be flooded for the first time in the third year after completion of construction?

(2) What is the probability of flooding in the first 3 years?

(3) What is the probability of flooding in 3 of the first 5 years?

(4) What is the probability of only one flood within 3 years?

ANS (1) 0.081 (2) 0.271 (3) 0.0081 (4) 0.243

3-18 On the average 2 damaging earthquakes occur in a certain area every 5 years. Assume the occurrence of earthquakes is a Poisson process in time.

(1) Determining the probability of getting 1 damaging earthquake in 3 years.

(2) Determine the probability of no earthquakes in 3 years.

(3) What is the probability of having at most 2 earthquakes in one year?

(4) What is the probability of having at least 1 earthquake in 5 years?

ANS (1) 0.3614 (2) 0.3011 (3) 0.9920 (4) 0.8647

3-19 Traffic on a one-way street that lead to a toll bridge is to be studied. The volume of the traffic is found to be 120 vehicles per hour *on the average* and out of which 2/3 are passenger cars and 1/3 are trucks. The toll at the bridge is $0.50 per car and $2 per truck. Assume that the arrivals of vehicles constitute a Poisson process.

(1) What is the probability that in a period of 1 minute, more than 3 vehicles will arrive at the toll bridge?

(2) What is the *expected* total amount of toll collected at the bridge in a period of 3 hours?

ANS (1) 0.1431 (2) $360.00

3-20 Steel construction work on multistorey building is a potentially hazardous occupation. A building contractor who is building a skyscraper at a steady pace finds that in spite of a strong emphasis on safety measures, he has been experiencing accidents among his large group of steel workers; on the average, about 1 accident occurs every 6 months.

(1) Assuming that the occurrence of a specific accident is not influenced by any previous accident, find the probability that there will be (exactly) 1 accident in the next 4 months.

(2) What is the probability of at least 1 accident in the next 4 months?

(3) What is the mean number of accidents that the contractor can expect in a year? What is the standard deviation for the number of accidents during a period of 1 year?

(4) If the contractor can go through a year without an accident among his steel construction workers, he will qualify for a safety award. What is the probability of his receiving this award next year?

(5) If the contractor's work is to continue at the same pace over the next 5 years, what is the probability that he will win the safety award twice during this 5-year period?

ANS (1) 0.3423 (2) 0.4866 (3) 2.0, 1.414 (4) 0.1353 (5) 0.1184

3-21 The occurrences of flood may be modelled by a Poisson process. The mean occurrence rate of floods for a certain region A is once every 8 years. Assume statistical independence between floods. A structure is located in region A. The probability that it will be inundated, when a flood occurs, is 0.05.

(1) Determine the probability of no flood in a 10-year period; of 1 flood; of more than 3 floods.

(2) Compute the probability that the structure will survive if there are no floods; if there is 1 flood; if there are n floods.

(3) Determine the probability that the structure will survive over 10-year period.

ANS (1) P(No flood in 10 years) = 0.2865, P(One flood in 10 years) = 0.3581,

P(More than three floods in 10 years) = 0.0383 (3) 0.9394

3-22 In a "torture test" a light switch is turned on and off until it fails. If the probability that a switch will fail any time it is turned on or off is 0.001, what is the probability that the switch will fail after it has been turned on or off 1,200 times. ANS 0.301

Continuous Probability Distributions

3-23 If the annual precipitation X in a city is a normally distributed random variable with a mean of 50 inch and a coefficient of variation of 0.2, determine the following:

(1) The standard deviation of X;

(2) $P(X < 30)$;

(3) $P(X > 60)$;

(4) $P(40 < X < 55)$;

(5) Probability that X is within 5 inch from the mean annual precipitation;

(6) The value x_0 such that the probability of the annual precipitation exceeding x_0 is only 1/4 that of not exceeding x_0.

ANS (1) 10 (2) 0.0228 (3) 0.1587 (4) 0.5328 (5) 0.3830 (6) 58.4

3-24 The present air traffic volume at an airport (number of landings and takeoffs) during the peak hour is a normally distributed random variable with a mean of 200 and a standard deviation of 60 airplanes.

(1) If the present runway capacity (for landings and takeoffs) is 350 planes per hour, what is currently the daily probability of air traffic congestion? Assume there is one peak hour daily.

(2) If no additional airports or expansion is built, what would be the probability of congestion 10 years hence? Assume that the mean traffic volume is increasing linearly at 10% of current volume per year, and the coefficient of variation remains the same.

(3) If the projected growth is correct, what airport capacity will be required 10 years from now to maintain the present service condition (that is, the same probability of congestion as now)?

ANS (1) 0.0062 (2) 0.6628 (3) 700

3-25 The depth to which a pile can be driven without hitting the rock stratum is denoted as H. For a certain construction site, suppose that this depth has a log-normal distribution with mean of 30 ft and coefficient of variation of 0.2. In order to provide satisfactory support, a pile should be embedded at least 1 ft into the rock stratum.

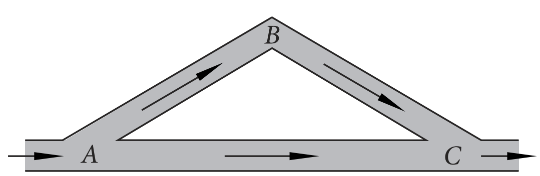
(1) What is the probability that a pile of length 40 ft will not anchor satisfactorily in rock?

(2) Suppose a 40-ft pile has been driven 39 ft into the ground and rock has not yet been encountered. What is the probability that an additional 5 ft of pile welded to the original length will be adequate to anchor this pile satisfactorily in rock?

ANS (1) 0.0778 (2) 0.707

3-26 A water distribution subsystem consists of pipes AB, BC, and AC as shown. Because of differences in elevation and in hydraulic head loss in the pipes and associated uncertainties, the capacity of each pipe (which is defined as the maximum rate of flow) is given as follows, in cfs (cubic feet per second):

- AB: capacity is normal with mean 5, coefficient of variation 0.1
- BC: capacity is log-normal with mean 5, coefficient of variation 0.1
- AC: capacity equal to 8 or 9 with equal likelihood



(1) Determine the probability that the capacity of the branch ABC will exceed 4 cfs.

(2) Determine the probability that the total capacity of the subsystem shown above will exceed 13 cfs. (Hint: Use conditional probability.) ANS (1) 0.9632 (2) 0.6016

3-27 The time of operation of a construction equipment until breakdown follows an exponential distribution with a mean of 24 months. The present inspection program is scheduled at every 5 months.

(1) What is the probability that an equipment will need repair at the first scheduled inspection date?

(2) If an equipment has not broken down by the first scheduled inspection date, what is the probability that it will be operational beyond the next scheduled inspection date?

(3) The company owns 5 pieces of this type of construction equipment; assuming that the service lives of equipments are statistically independent, determine the probability that at most 1 piece of equipment will need repair at the scheduled inspection date?

(4) If it is desired to limit the probability of repair at each scheduled inspection date to not more than 10%, what should be the inspection interval? The conditions of part (3) remain valid.

ANS (1) 0.188 (2) 0.812 (3) 0.762 (4) 0.506 month

Sampling Distribution and Confidence Intervals

3-28 A structure is designed to rest on 100 piles. Nine test piles were driven at random locations into the supporting soil stratum and loaded until failure occurred; the results of pile capacity test (in tons) are as follows:

82, 75, 95, 90, 88, 92, 78, 85, 80

(1) Estimate the mean and standard deviation of the individual pile capacity to be used at the site.

(2) Establish the 98% confidence interval for the mean pile capacity, assuming known $\sigma = s$.

(3) Determine the 98% confidence interval for the mean pile capacity on the basis of unknown variance.

ANS (1) mean = 85.0 tons, standard deviation = 6.764 (2) (79.76, 90.24) (3) (78.47, 91.53)

3-29 The following sample data were obtained for the time (in minutes) required to drive a pile into foundation:

11.2, 18.5, 9.9, 12.9, 10.2, 10.5, 8.7, 12.4, 13.5, 15.4, 12.6, 16.7, 14.4, 17.7, 15.5

(1) Determine the 95% confidence interval of the mean pile-driving time.

(2) Establish the 95% confidence interval of the standard deviation in pile-driving time.

ANS (1) (11.688, 14.992) (2) (2.174, 4.728)

3-30 (1) In a traffic survey where speeds of vehicles are measured, it is desired to determine the mean vehicle speed to within ± 1 kph (kilometre per hour) with 99% confidence. From a preliminary study, the standard deviation of the vehicle speed is found to be 3.58 kph. Assume that all observations are independent; determine the number of observations required.

(2) If 150 observations were taken, what would be the confidence level associated with the interval of ± 1 kph of the mean speed? Assume that the standard deviation of vehicle speed is still 3.58 kph.

ANS (1) 86 (2) 0.9994

3-31 From a set of data on the daily BOD level at a certain station for 30 days, the following have been computed: $\bar{X} = 3.5$ (mg/L), $s^2 = 0.184$ (mg/L)^2.

(1) Determine the 99.5% confidence interval for the mean BOD.

(2) If the engineer is not satisfied with the width of the confidence interval established in part (1), and would like to reduce this interval by 10%, keeping the 99.5% confidence level, how many *additional* daily measurements have to be gathered?

ANS (1) (3.262, 3.738) (2) 36

Linear Regression

3-32 The following data were obtained from a stress test on rods fabricated from an experimental alloy:

Lateral Strain Y	Longitudinal Strain X
0.0006	0.002
0.0011	0.003
0.0014	0.004
0.0016	0.005
0.0022	0.006
0.0035	0.010
0.0046	0.015
0.0069	0.020
0.0086	0.025
0.0102	0.030

$$\sum_{i=1}^{10} y_i = 0.0407, \quad \sum_{i=1}^{10} x_i = 0.120, \quad \sum_{i=1}^{10} x_i y_i = 0.0007943, \quad \sum_{i=1}^{10} y_i^2 = 0.00027, \quad \sum_{i=1}^{10} x_i = 0.00234$$

(1) Use the method of least square to determine the estimated regression equation for predicting lateral strain from longitudinal strain.

(2) Poisson's ratio is the lateral strain divided by the longitudinal strain. If the true Y-intercept is zero, what is the estimated value of this quantity provided by your regression results?

(3) Compute the sample standard deviation for the lateral strain Y.

(4) Compute the standard error of the estimate for lateral strain Y.

(5) What is the estimate of $\sigma^2_{Y|X}$?

(6) Determine the predicted lateral strain when the longitudinal strain is 0.012.

(7) Construct a 99% confidence interval for the predicted lateral strain when the longitudinal strain is 0.012.

(8) Construct a 99% confidence interval for the mean lateral strain when the longitudinal strain is 0.012.

(9) Determine the sample correlation coefficient.

ANS (1) $a = -0.00001$, $b = 0.3400$ (2) 0.3392 (3) 0.003406 (4) 0.0002219 (5) 4.925×10^{-8} (6) 0.00407 (7) $(0.00329, 0.00485)$ (8) $(0.00383, 0.00431)$ (9) 0.9979