MATHEMATICS AND MODELLING REFRESHER COURSE PRACTICE PROBLEMS

Calculus

Differentiation

2-1 For $y = x^3 + \dfrac{7}{x^4} - \dfrac{2}{x} + 12$, find y'. ANS $3x^2 - \dfrac{28}{x^5} + \dfrac{2}{x^2}$

2-2 For $y = 5x^3 - 2^x + 3e^x$, find y'. ANS $15x^2 - 2^x \ln 2 + 3e^x$

2-3 For $y = 2\tan x + \sec x - 1$, find y'. ANS $\sec x(2\sec x + \tan x)$

2-4 For $y = \sin x \cdot \cos x$, find y'. ANS $\cos 2x$

2-5 For $y = x^2 \ln x$, find y'. ANS $x(2\ln x + 1)$

2-6 For $y = 3e^x \cos x$, find y'. ANS $3e^x(\cos x - \sin x)$

2-7 For $y = \dfrac{\ln x}{x}$, find y'. ANS $\dfrac{1 - \ln x}{x^2}$

2-8 For $y = \dfrac{e^x}{x^2} + \ln 3$, find y'. ANS $\dfrac{e^x(x-2)}{x^3}$

2-9 For $y = 2x^2 + \ln x$, find y''. ANS $4 - \dfrac{1}{x^2}$

2-10 For $y = e^{2x-1}$, find y''. ANS $4e^{2x-1}$

2-11 For $y = x\cos x$, find y''. ANS $-2\sin x - x\cos x$

2-12 For $y = e^{-x}\sin x$, find y''. ANS $-2e^{-x}\cos x$

2-13 For $y = \tan x$, find y''. ANS $2\sec^2 x \tan x$

2-14 For $y = \dfrac{e^x}{x}$, find y''. ANS $\dfrac{e^x(x^2 - 2x + 2)}{x^3}$

2-15 Find the Taylor series at $x_0 = 0$ (MacLaurin series) of $f(x)y(x) = \cos x$.

ANS $$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} + \cdots$$

2-16 Find the Taylor series at $x_0 = 0$ (MacLaurin series) of $f(x) = \ln(1+x)$.

ANS $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + (-1)^{n-1}\frac{1}{n}x^n + \cdots$

Integration

2-17 Evaluate $\int x\sqrt{x}\,dx$. ANS $\frac{2}{5}x^{5/2} + C$

2-18 Evaluate $\int \frac{(1-x)^2}{\sqrt{x}}\,dx$. ANS $\frac{2\sqrt{x}\left(15-10x+3x^2\right)}{15} + C$

2-19 Evaluate $\int \frac{x^2}{1+x^2}\,dx$. ANS $x - \tan^{-1}x + C$

2-20 Evaluate $\int \frac{dx}{\sqrt[3]{2-3x}}$. ANS $-\frac{1}{2}(2-3x)^{2/3} + C$

2-21 Evaluate $\int \frac{dx}{x\ln x\ln\ln x}$. ANS $\ln\left|\ln\ln x\right| + C$

2-22 Evaluate $\int \frac{dx}{e^x + e^{-x}}$. ANS $\tan^{-1}e^x + C$

2-23 Evaluate $\int \cos^3 x\,dx$. ANS $\sin x - \frac{1}{3}\sin^3 x + C$

2-24 Evaluate $\int \sin 2x\cos 3x\,dx$. ANS $\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

2-25 Evaluate $\int \frac{dx}{x\sqrt{x^2-1}}$. ANS $\cos^{-1}\frac{1}{|x|} + C$

2-26 Evaluate $\int \frac{dx}{1+\sqrt{2x}}$. ANS $\sqrt{2x} - \ln\left(1+\sqrt{2x}\right) + C$

2-27 Evaluate $\int x^2\ln x\,dx$. ANS $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$

2-28 Evaluate $\int x^2\tan^{-1}x\,dx$. ANS $\frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$

2-29 Evaluate $\int x\sin x\cos x\,dx$. ANS $-\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$

2-30 Evaluate $\int x\ln(x-1)\,dx$. ANS $\frac{1}{2}(x^2-1)\ln(x-1) - \frac{1}{4}x^2 - \frac{1}{2}x + C$

2-31 Evaluate $\int \frac{x^3}{x+3}\,dx$. ANS $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln\left|x+3\right| + C$

2-32 Evaluate $\displaystyle\int \frac{2x+3}{x^2+3x-10}\,\mathrm{d}x$. ANS $\ln|x-2| + \ln|x+5| + C$ or $\ln|x^2+3x-10| + C$

2-33 Evaluate $\displaystyle\int \frac{x^5+x^4-8}{x^3-x}\,\mathrm{d}x$. ANS $\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 8\ln|x| - 4\ln|x+1| - 3\ln|x-1| + C$

2-34 Evaluate $\displaystyle\int \frac{x^2+1}{(x+1)^2(x-1)}\,\mathrm{d}x$. ANS $\dfrac{1}{x+1} + \frac{1}{2}\ln|x^2-1| + C$

2-35 Evaluate $\displaystyle\int \frac{1}{(x^2+1)(x^2+x)}\,\mathrm{d}x$. ANS $\ln|x| - \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1}x + C$

2-36 Evaluate $\displaystyle\int \frac{1}{(x^2+1)(x^2+x+1)}\,\mathrm{d}x$. ANS $-\frac{1}{2}\ln\dfrac{x^2+1}{x^2+x+1} + \dfrac{\sqrt{3}}{3}\tan^{-1}\dfrac{2x+1}{\sqrt{3}} + C$

2-37 Evaluate $\displaystyle\int_4^9 \sqrt{x}\left(1+\sqrt{x}\right)\mathrm{d}x$. ANS $\frac{271}{6}$

2-38 Evaluate $\displaystyle\int_0^{2\pi} |\sin x|\,\mathrm{d}x$. ANS 4

2-39 Evaluate $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{8-2x^2}\,\mathrm{d}x$. ANS $\sqrt{2}(\pi+2)$

2-40 Determine the area A of the region bounded by two parabolas $y=x$ and $y=x^2$. ANS $\frac{1}{3}$

2-41 Determine the area A of the region bounded by $y=\dfrac{1}{x}$, $y=x$, and $x=2$. ANS $\frac{3}{2} - \ln 2$

2-42 Determine the area A of the region bounded by $y=x^2$ and $y=2x+3$. ANS $\frac{32}{3}$

Partial Derivatives

2-43 For $z = \sin(xy) + \cos^2(xy)$, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

ANS $\dfrac{\partial z}{\partial x} = y\big[\cos(xy) - \sin(2xy)\big]$, $\dfrac{\partial z}{\partial y} = x\big[\cos(xy) - \sin(2xy)\big]$

2-44 For $z = \ln\tan\dfrac{x}{y}$, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. ANS $\dfrac{\partial z}{\partial x} = \dfrac{2}{y}\csc\dfrac{2x}{y}$, $\dfrac{\partial z}{\partial y} = -\dfrac{2x}{y^2}\csc\dfrac{2x}{y}$

2-45 For $z = x^4+y^4-4x^2y^2$, determine $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$, and $\dfrac{\partial^2 z}{\partial x\partial y}$.

ANS $\dfrac{\partial^2 z}{\partial x^2} = 12x^2-8y^2$, $\dfrac{\partial^2 z}{\partial y^2} = 12y^2-8x^2$, $\dfrac{\partial^2 z}{\partial x\partial y} = -16xy$

2-46 For $z = \tan^{-1}\dfrac{y}{x}$, determine $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$, and $\dfrac{\partial^2 z}{\partial x \partial y}$.

ANS $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{2xy}{(x^2+y^2)^2}$, $\quad \dfrac{\partial^2 z}{\partial y^2} = -\dfrac{2xy}{(x^2+y^2)^2}$, $\quad \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

2-47 For $z = x\ln(xy)$, determine $\dfrac{\partial^3 z}{\partial x^2 \partial y}$ and $\dfrac{\partial^3 z}{\partial x \partial y^2}$. ANS $\dfrac{\partial^3 z}{\partial x^2 \partial y} = 0$, $\quad \dfrac{\partial^3 z}{\partial x \partial y^2} = -\dfrac{1}{y^2}$

2-48 For $z = u^2 + v^2$, $u = x + y$, $v = x - y$, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. ANS $\dfrac{\partial z}{\partial x} = 4x$, $\quad \dfrac{\partial z}{\partial y} = 4y$

2-49 For $z = \mathrm{e}^u \sin v$, $u = xy$, $v = x + y$, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

ANS $\dfrac{\partial z}{\partial x} = \mathrm{e}^{xy}\big[y\sin(x+y) + \cos(x+y)\big]$, $\quad \dfrac{\partial z}{\partial y} = \mathrm{e}^{xy}\big[x\sin(x+y) + \cos(x+y)\big]$

2-50 For $z = u^2 \ln v$, $u = \dfrac{x}{y}$, $v = 3x - 2y$, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

ANS $\dfrac{\partial z}{\partial x} = \dfrac{2x}{y^2}\ln(3x-2y) + \dfrac{3x^2}{(3x-2y)\,y^2}$, $\quad \dfrac{\partial z}{\partial y} = -\dfrac{2x^2}{y^3}\ln(3x-2y) - \dfrac{2x^2}{(3x-2y)\,y^2}$

2-51 For $\sin y + \mathrm{e}^x - xy^2 = 0$, determine $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. ANS $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2 - \mathrm{e}^x}{\cos y - 2xy}$

2-52 For $z = uv + \sin t$, $u = \mathrm{e}^t$, $v = \cos t$, determine $\dfrac{\mathrm{d}z}{\mathrm{d}t}$. ANS $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{e}^t(\cos t - \sin t) + \cos t$

2-53 For $x + 2y + z - 2\sqrt{xyz} = 0$, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

ANS $\dfrac{\partial z}{\partial x} = \dfrac{yz - \sqrt{xyz}}{\sqrt{xyz} - xy}$, $\quad \dfrac{\partial z}{\partial y} = \dfrac{xz - 2\sqrt{xyz}}{\sqrt{xyz} - xy}$

2-54 For $x^2 + y^2 + z^2 - 4z = 0$, determine $\dfrac{\partial^2 z}{\partial x^2}$. ANS $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{(2-z)^2 + x^2}{(2-z)^3}$

Multiple Integrals

2-55 Evaluate the double iterated integral $I = \displaystyle\int_0^1 \int_0^{\tan x} \frac{1}{1+y^2}\,\mathrm{d}y\,\mathrm{d}x$. ANS $\frac{1}{2}$

2-56 Evaluate the double iterated integral $I = \displaystyle\int_0^1 \int_0^{x} \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y\,\mathrm{d}x$. ANS $\frac{1}{2}\pi - 1$

2-57 Evaluate $\displaystyle\iint_{\mathcal{R}} x\,\mathrm{d}A$, where $\mathcal{R}$ is bounded by $y = 3x$, $y = x$, $x + y = 4$. ANS 2

2-58 Evaluate the double iterated integral by reversing the order of integration:

$$I = \int_{-2}^{0}\int_{-2}^{x} \frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}y\mathrm{d}x. \qquad \text{ANS } 2(\sqrt{2}-1)$$

2-59 Evaluate the double iterated integral by reversing the order of integration:

$$I = \int_{0}^{2}\int_{0}^{\sqrt{4-x^2}} (4-y^2)^{3/2}\,\mathrm{d}y\mathrm{d}x. \qquad \text{ANS } \frac{256}{15}$$

2-60 Evaluate the double iterated integral

$$I = \int_{0}^{1}\int_{0}^{1} \left|x-y\right|\mathrm{d}y\mathrm{d}x. \qquad \text{ANS } \frac{1}{3}$$

2-61 Evaluate the double integral $\iint_{\mathcal{R}} \left|y-2x^2+1\right|\mathrm{d}A$, where $\mathcal{R}$ is the square bounded by $x=\pm 1$, $y=\pm 1$.

ANS $\frac{44}{15}$

2-62 Find the area bounded by the curves $y=\dfrac{x^2+1}{x+1}$, $x+3y=7$. ANS $\frac{35}{6}-2\ln 6$

2-63 Evaluate the triple integral $\iiint_V (xy+z)\,\mathrm{d}V$ over the volume bounded by $y+z=1$, $z=2y$, $z=y$, $x=0$, $x=3$. ANS $\frac{29}{144}$

2-64 Find the volume of the region bounded by the surfaces $z=16-x^2-4y^2$, $x+y=1$, $z=16$, $x=0$, $y=0$ (in the first octant). ANS $\frac{5}{12}$

Vector Calculus

2-65 Find the functions $f(x,y)$ such that $\nabla f = \operatorname{grad} f = (3x^2y^2+3)\,\hat{\mathbf{i}} + (2x^3y+2)\,\hat{\mathbf{j}}$.

ANS $f = x^3y^2 + 3x + 2y + C$

2-66 Find all functions $f(x,y,z)$ such that ∇f is equal to the vector field

$$\mathbf{F}(x,y,z) = (3x^2y+yz+2xz^2)\,\hat{\mathbf{i}} + (xz+x^3+3z^2-6y^2z)\,\hat{\mathbf{j}} + (2x^2z+6yz-2y^3+xy)\,\hat{\mathbf{k}}.$$

ANS $f = x^3y + xyz + x^2z^2 + 3yz^2 - 2y^3z + C$

2-67 Using the Divergence Theorem, evaluate the surface integral

$$\oiint_S (x^2\,\hat{\mathbf{i}} + y^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

where S is the surface enclosing the volume bounded by $x = 0,\ y = 0,\ z = 0,\ x = a,\ y = a,\ z = a$, and $\hat{\mathbf{n}}$ is the unit *outward* normal to S. ANS $3a^4$

2-68 Using the Divergence Theorem, evaluate the surface integral

$$\oiint_S (4xz\,\hat{\mathbf{i}} - y^2\,\hat{\mathbf{j}} + yz\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

where S is the surface enclosing the volume bounded by $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1$, and $\hat{\mathbf{n}}$ is the unit *outward* normal to S. ANS $\frac{3}{2}$