UNENE Math Refresher Course

Algebra

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Exponential Functions

$y = a^x$ is an exponential function.

Properties

1. $a^0 = 1, \quad a \neq 0$

2. $a^m \cdot a^n = a^{m+n}$

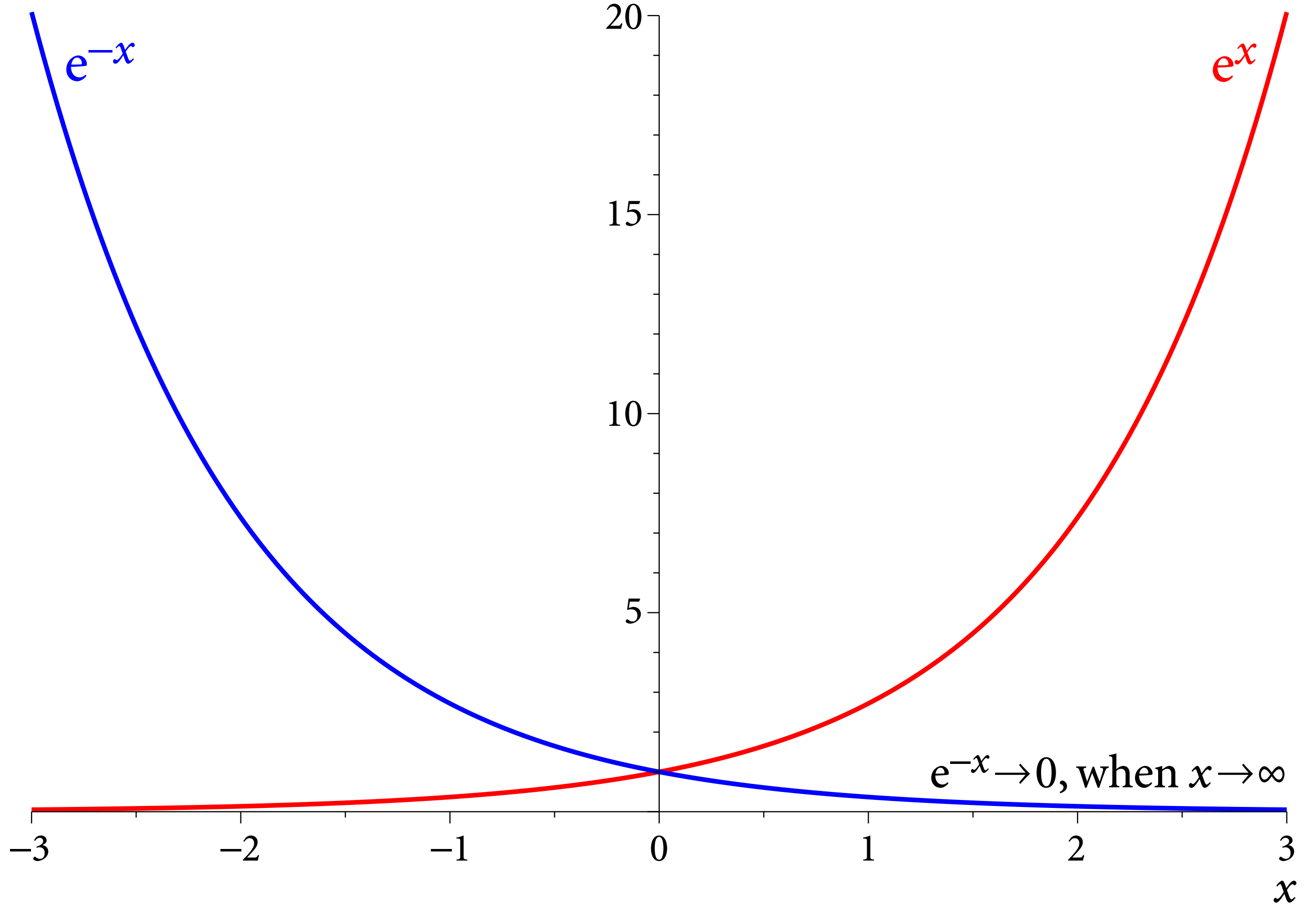
3. $\dfrac{1}{a^n} = a^{-n}, \quad \dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$

4. $(a^m)^n = a^{m \cdot n}$

5. $(a\,b)^n = a^n \cdot b^n$

6. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

For the special case $y = e^x$, $e = 2.718281828\ldots$



Logarithmic Functions

If $x = a^y, \quad a > 0, \quad a \neq 1 \quad \Longrightarrow \quad y = \log_a x$, where a is called the base.

Properties

1. $a = a^1 \quad \Longrightarrow \quad \log_a a = 1$

2. $1 = a^0 \quad \Longrightarrow \quad \log_a 1 = 0$

3. $\log_a b^x = x \log_a b$

4. $\log_a x + \log_a y = \log_a (xy)$

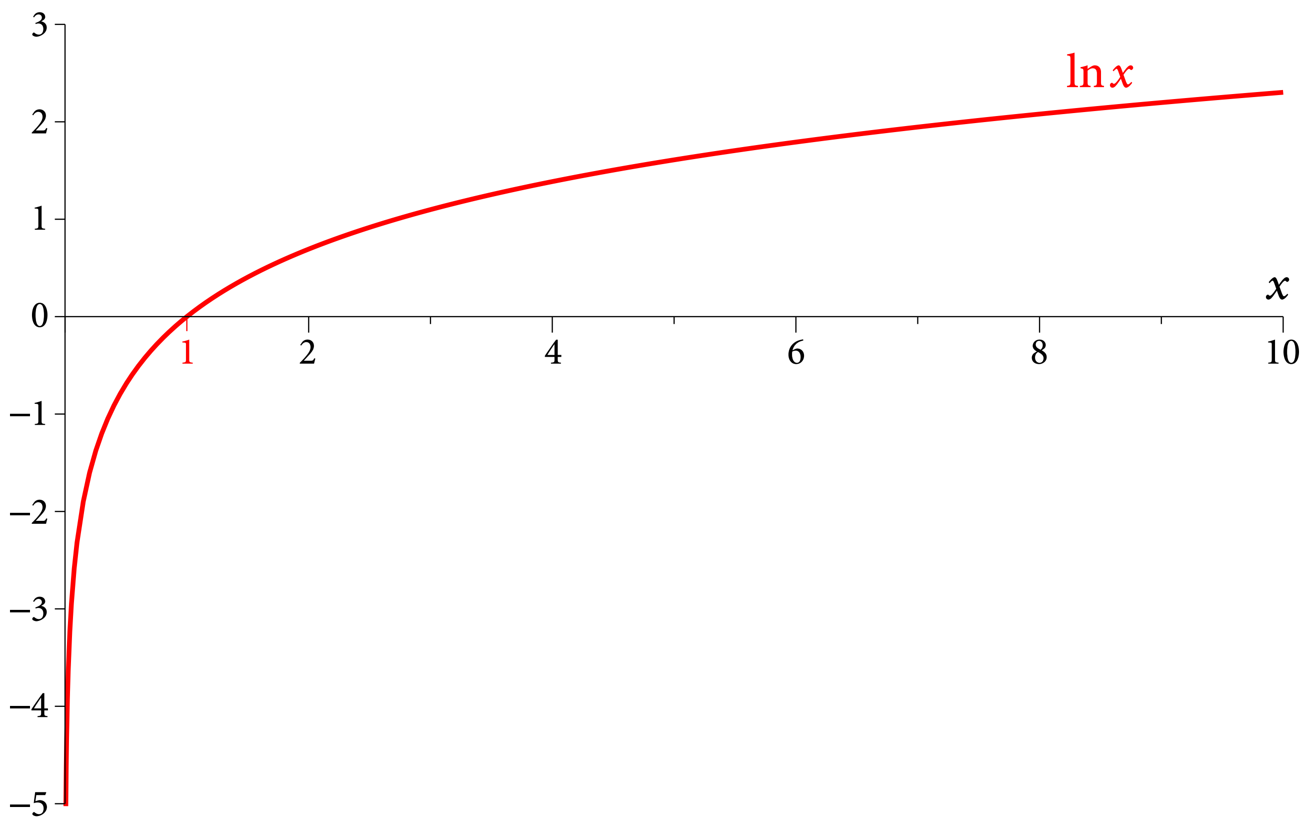
5. $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$

6. $a^{\log_a x} = x$

7. $\log_a x = \dfrac{\log_b x}{\log_b a}$

In mathematics and engineering, the most frequently used bases: $a = 10$ and e.

- $a = 10:$ $\quad \log_{10} x = \lg x$
- $a = \mathrm{e}:$ $\quad \log_{\mathrm{e}} x = \ln x$



Example

Simplify $\quad y = \dfrac{\ln x^2 - \ln \dfrac{1}{x}}{\ln \sqrt[3]{x}}.$

$$y = \frac{\ln x^2 - \ln x^{-1}}{\ln x^{\frac{1}{3}}} = \frac{2 \ln x - (-\ln x)}{\frac{1}{3} \ln x} = \frac{3 \ln x}{\frac{1}{3} \ln x} = 9$$

Example

Solve for x in equation $20 = 500\left(1 - \dfrac{4}{4 + e^{-0.002x}}\right)$.

Divide the equation by 500: $\dfrac{1}{25} = 1 - \dfrac{4}{4 + e^{-0.002x}}$

Simplify: $\dfrac{4}{4 + e^{-0.002x}} = \dfrac{24}{25} \implies 4 + e^{-0.002x} = \dfrac{25}{6}$

$$e^{-0.002x} = \frac{1}{6}$$

Take ln of both sides: $\ln e^{-0.002x} = \ln\dfrac{1}{6} \implies -0.002x = -\ln 6$

$$\therefore \quad x = \frac{1}{0.002}\ln 6 = 895.8797$$

Example

- Given that $y = y_0 e^{-ax^b}$, $x > 0$, solve for x.
- If $y = 0.1$, $y_0 = 10.2$, $a = 0.5$, $b = 2.1$, evaluate x.

- Taking ln of both sides of the equation

$$\ln y = \ln y_0 + \ln e^{-ax^b} = \ln y_0 - ax^b$$

$$\therefore \quad x^b = \frac{\ln y_0 - \ln y}{a} \quad \Longrightarrow \quad x = \left(\frac{\ln y_0 - \ln y}{a} \right)^{\frac{1}{b}}$$

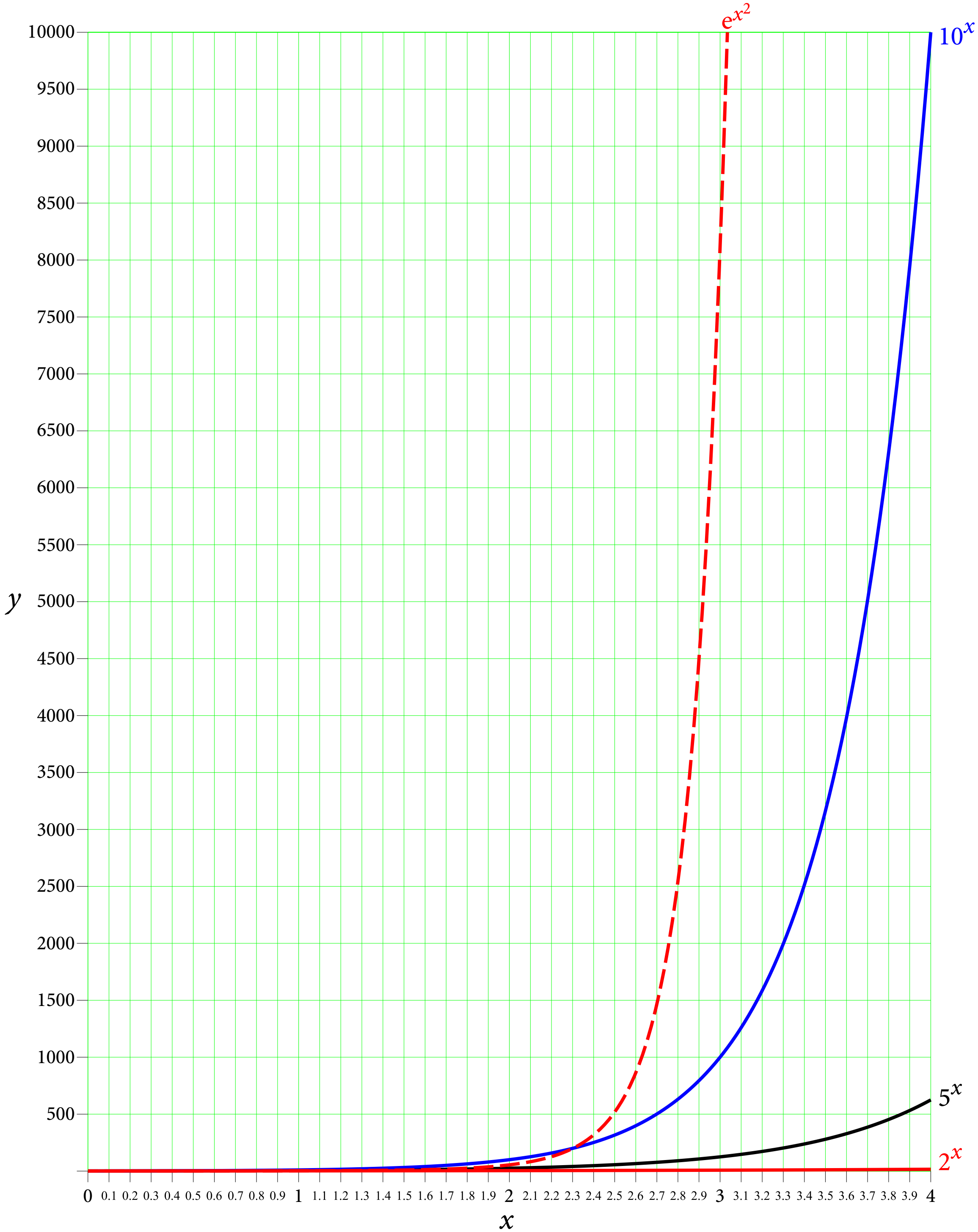
- For $y = 0.1$, $y_0 = 10.2$, $a = 0.5$, $b = 2.1$

$$x = \left(\frac{\ln 10.2 - \ln 0.1}{0.5} \right)^{\frac{1}{2.1}} = 9.2499^{\frac{1}{2.1}} = 2.88$$

Graphs of Functions

The following figure shows a plot of functions $y = 2^x,\ 5^x,\ 10^x,\ e^{x^2}$.

It is obvious that we cannot see any detail of 2^x for all values of x and not much details of $5^x,\ 10^x,\ e^{x^2}$ for $x < 2$.


e^{x^2}
10^x
5^x
2^x
y
x
10000
9500
9000
8500
8000
7500
7000
6500
6000
5500
5000
4500
4000
3500
3000
2500
2000
1500
1000
500
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4

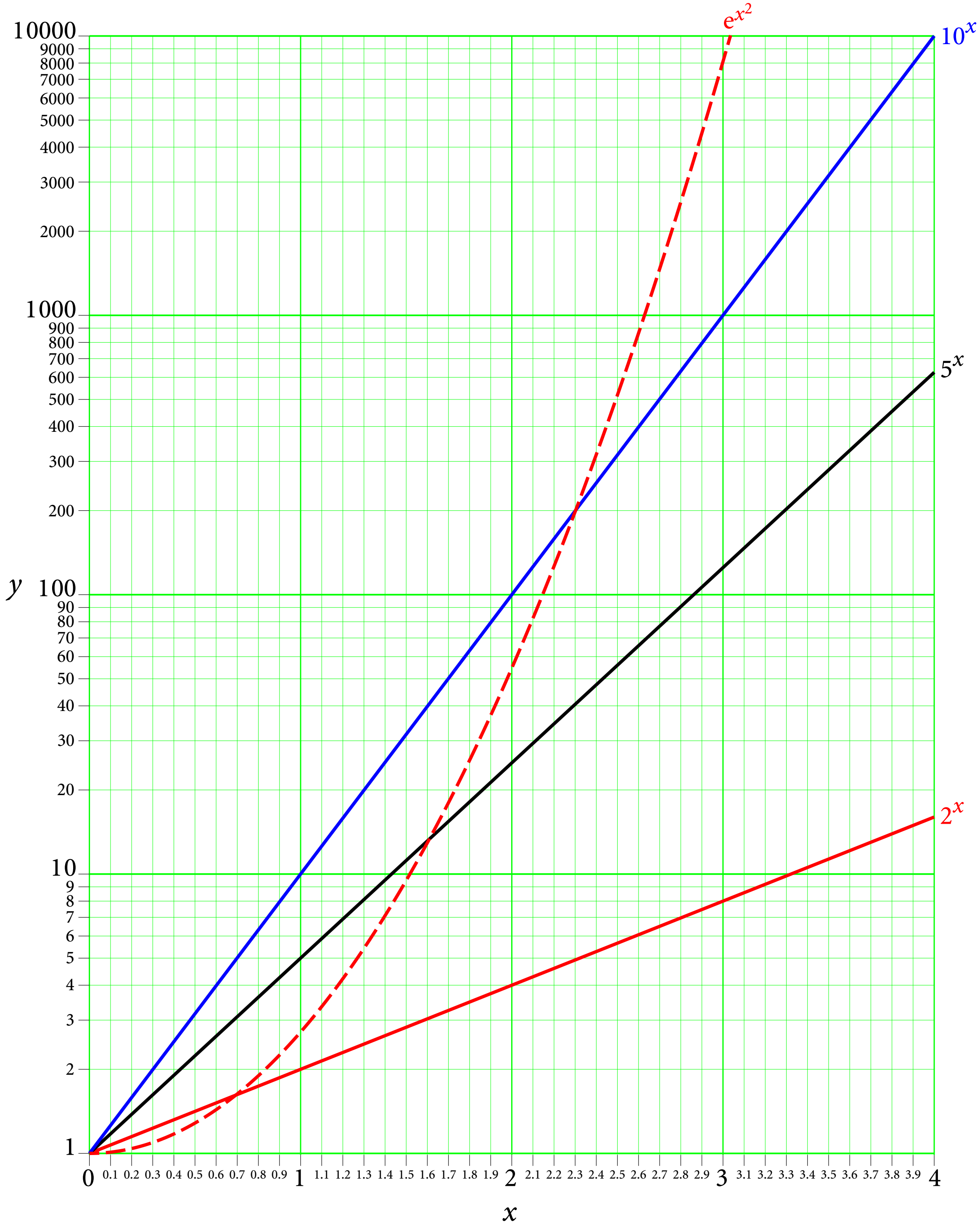
Semi-Log Graphs

The same functions plotted using a semi-logarithmic scale. Details are revealed for all values of x.

- For semi-log plots, the numbers along the horizontal x-axis are (linearly) evenly spaced, while along the vertical y-axis, powers of 10 are evenly spaced.
- For exponential function $y = a^x, \quad a > 0$, taking logarithm of both sides gives

$$\log_{10} y = x \cdot \log_{10} a$$

The function appears as a straight line when plotted on semi-log paper.


e^{x^2}
10^x
5^x
2^x
y
x
10000
1000
100
10
1
0
1
2
3
4

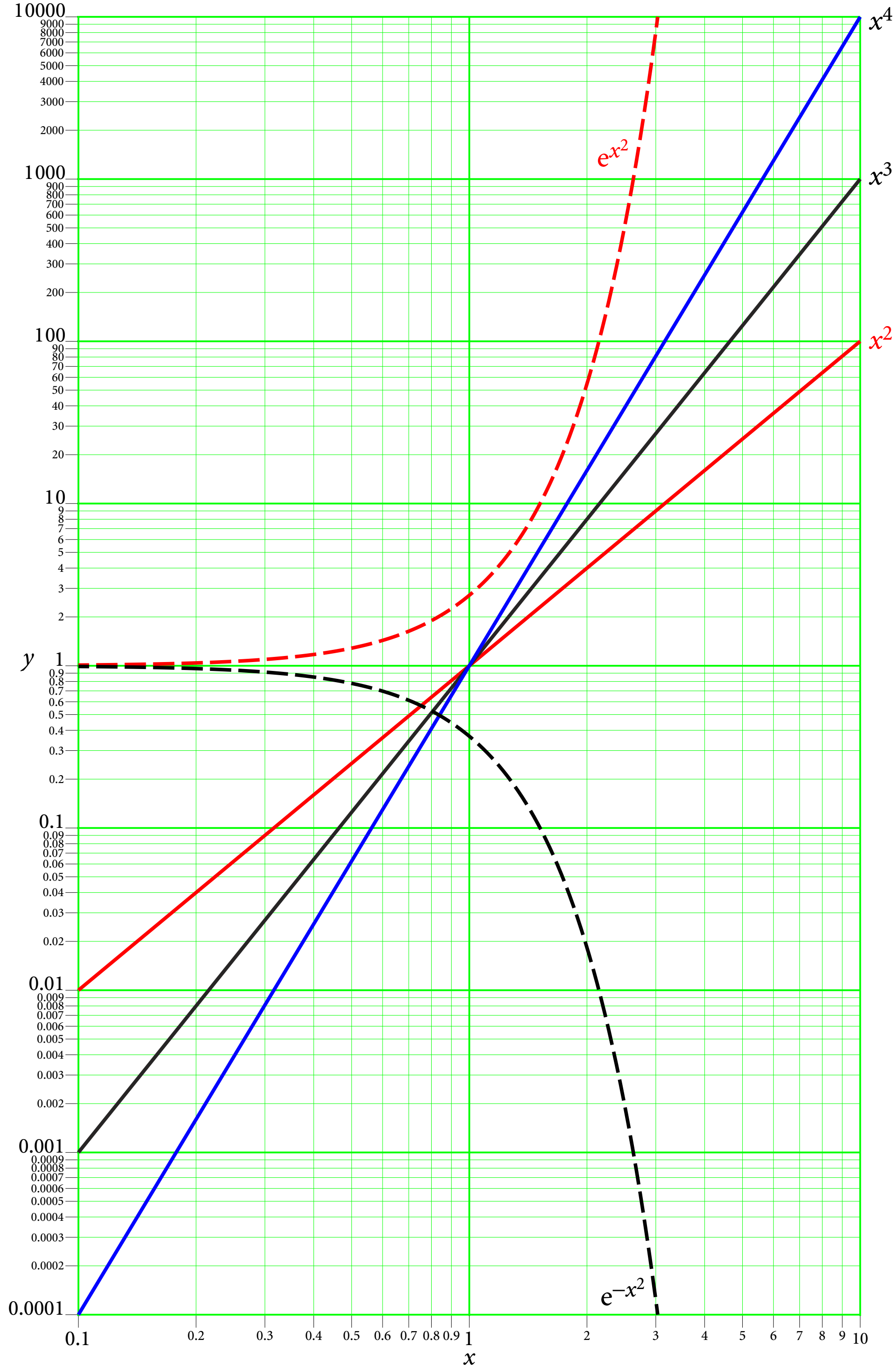
Log-Log Graphs

The following figure shows a plot of functions $y = x^2,\ x^3,\ x^4,\ e^{x^2},\ e^{-x^2}$ plotted using the logarithmic scales (both axes using log scales), in which powers of 10 are evenly spaced.

- For function $y = x^a, \quad x > 0$, taking logarithm of both sides gives

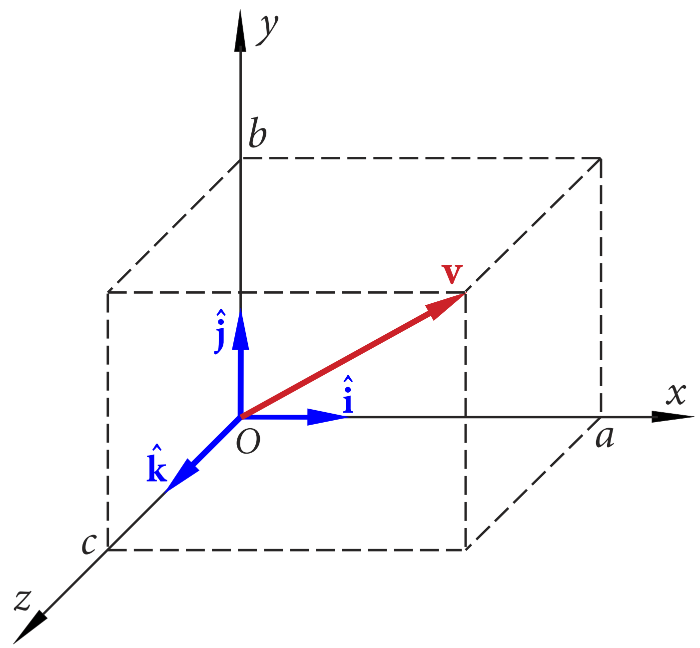
$$\log_{10} y = a \cdot \log_{10} x$$

  The function appears as a straight line when plotted on log-log paper.


x^4
x^3
x^2
e^{x^2}
e^{-x^2}
y
x
10000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5
6
7
8
9
10

Vectors

Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ be the unit vectors in the x-, y-, z-directions, respectively.



- A vector $\mathbf{v}$ is $\mathbf{v} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$

  ☞ A vector is printed in boldface $\mathbf{v}$, or written as $\vec{v}$ in handwriting.

- The norm (length) of a vector is $v = |\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$

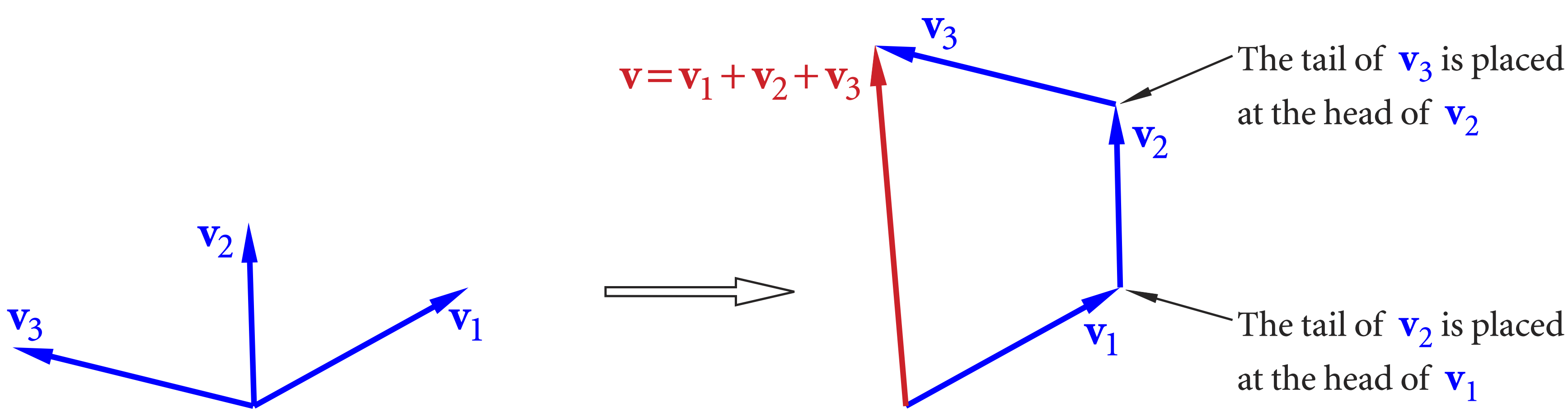
- The unit vector in direction $\mathbf{v}$ is $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Vector Addition

- Let $\mathbf{v}_1 = a_1\,\hat{\mathbf{i}} + b_1\,\hat{\mathbf{j}} + c_1\,\hat{\mathbf{k}}, \quad \mathbf{v}_2 = a_2\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + c_2\,\hat{\mathbf{k}}.$

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1\\ &= (a_1\,\hat{\mathbf{i}} + b_1\,\hat{\mathbf{j}} + c_1\,\hat{\mathbf{k}}) + (a_2\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + c_2\,\hat{\mathbf{k}})\\ &= (a_1 + a_2)\,\hat{\mathbf{i}} + (b_1 + b_2)\,\hat{\mathbf{j}} + (c_1 + c_2)\,\hat{\mathbf{k}}\end{aligned}$$

- Graphically, the sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ is obtained by placing them head to tail and drawing the vector $\mathbf{v}$ from the free tail to the free head.



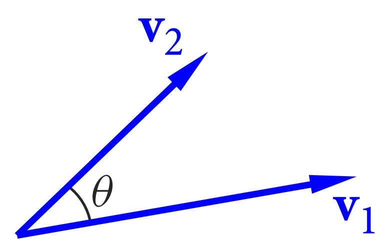
Dot Product of Vectors

- Let $\mathbf{v}_1 = a_1\,\hat{\mathbf{i}} + b_1\,\hat{\mathbf{j}} + c_1\,\hat{\mathbf{k}}$, $\mathbf{v}_2 = a_2\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + c_2\,\hat{\mathbf{k}}$.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (a_1\,\hat{\mathbf{i}} + b_1\,\hat{\mathbf{j}} + c_1\,\hat{\mathbf{k}}) \cdot (a_2\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + c_2\,\hat{\mathbf{k}})$$
$$= a_1\,a_2 + b_1\,b_2 + c_1\,c_2$$

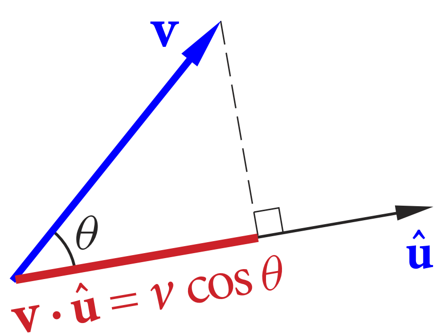
✍ A scalar

$$= |\mathbf{v}_1|\,|\mathbf{v}_2| \cos\theta$$



- The **projection** of vector $\mathbf{v}$ in a given direction, specified by the unit vector $\hat{\mathbf{u}}$, is given by

$$\mathbf{v} \cdot \hat{\mathbf{u}} = v \cos\theta$$



Cross Product of Vectors

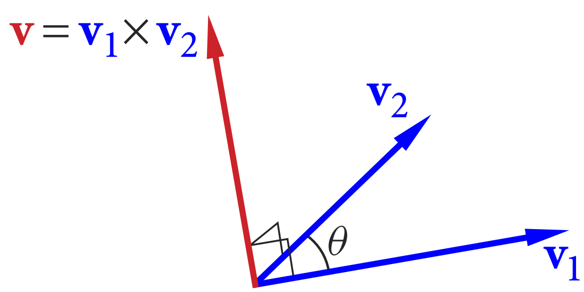
- Let $\mathbf{v}_1 = a_1\,\hat{\mathbf{i}} + b_1\,\hat{\mathbf{j}} + c_1\,\hat{\mathbf{k}}, \quad \mathbf{v}_2 = a_2\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + c_2\,\hat{\mathbf{k}}.$

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} \\ a_1 & b_1 \\ a_2 & b_2 \end{matrix}$$

$$= (b_1 c_2 - b_2 c_1)\,\hat{\mathbf{i}} + (c_1 a_2 - c_2 a_1)\,\hat{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\,\hat{\mathbf{k}}$$

✍ A vector

- **Direction** obtained using the right-hand rule: flatten the right hand, four fingers go along $\mathbf{v}_1$, then curl the fingers (palm) towards $\mathbf{v}_2$; the direction of the thumb is the direction of $\mathbf{v}_1 \times \mathbf{v}_2$.



- **Magnitude** $\quad |\mathbf{v}_1 \times \mathbf{v}_2| = |\mathbf{v}_1|\,|\mathbf{v}_2| \sin\theta$

Gaussian Elimination

Solve the following system of linear algebraic equations

$$3x_1 + 4x_2 = 10 \qquad (1)$$

$$2x_1 - 5x_2 = -1 \qquad (2)$$

- To solve for x_1, i.e., to eliminate x_2,

$$\begin{array}{lrcrcr l}
\text{Eqn. (1)} \times 5: & 15x_1 & + & 20x_2 & = & 50 & \\
\text{Eqn. (2)} \times 4: & 8x_1 & - & 20x_2 & = & -4 & (+ \\
\hline
 & 23x_1 & & & = & 46 & \Longrightarrow \quad x_1 = 2
\end{array}$$

- Similarly, to solve for x_2, i.e., to eliminate x_1,

$$\begin{array}{lrcrcr l}
\text{Eqn. (1)} \times 2: & 6x_1 & + & 8x_2 & = & 20 & \\
\text{Eqn. (2)} \times 3: & 6x_1 & - & 15x_2 & = & -3 & (- \\
\hline
 & & & 23x_2 & = & 23 & \Longrightarrow \quad x_2 = 1
\end{array}$$

Alternatively, having obtained x_1, x_2 can be found from either Eqn. (1) or (2).

From Eqn. (1): $$x_2 = \frac{10 - 3x_1}{4} = \frac{10 - 3 \times 2}{4} = 1$$

Consider a system of n linear algebraic equations

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n &= b_2 \\ \cdots\ \ \cdots& \\ a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}\,x_n &= b_n \end{aligned}$$

where $x_1, x_2, \ldots, x_n$ are the n unknowns.

The system can be written in the *matrix* form

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}}_{\mathbf{x}} = \underbrace{\begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}}_{\mathbf{b}} \implies \mathbf{A}\,\mathbf{x} = \mathbf{b}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{x}$ is the column vector of unknowns, and $\mathbf{b}$ is the column vector of right-hand side constants.

Operations of Matrices

Addition of Matrices

Let $\mathbf{A} = [\,a_{ij}\,]_{m\times n}$, $\mathbf{B} = [\,b_{ij}\,]_{m\times n}$. $\mathbf{A}$ and $\mathbf{B}$ must be the same size.

$\mathbf{C} = \mathbf{A} + \mathbf{B} \implies c_{ij} = a_{ij} + b_{ij}$

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, $\quad (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

Multiplication by a Scalar

Let $\mathbf{A} = [\,a_{ij}\,]_{m\times n}$. $\quad \mathbf{C} = \alpha\mathbf{A} \implies c_{ij} = \alpha\, a_{ij}$, α is a scalar.

$$\alpha \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \end{bmatrix}$$

Multiplication of Matrices

Let $\mathbf{A} = [\,a_{ik}\,]_{m\times n}$, $\mathbf{B} = [\,b_{kj}\,]_{n\times l}$

$$\mathbf{C} = \mathbf{AB} \quad \Longrightarrow \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$\begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_{1j} & \cdots \\ \cdots & b_{2j} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & b_{nj} & \cdots \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & c_{ij} & \cdots \\ & \vdots & \end{bmatrix}$$

ith row $\qquad$ jth column $\qquad$ ijth element

- **AB** is usually **not** the same as **BA.**

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \\ a_{31}b_{11}+a_{32}b_{21} & a_{31}b_{12}+a_{32}b_{22} \end{bmatrix}$$

Transpose of Matrices

Let $\mathbf{A} = [\, a_{ij} \,]_{m \times n} \quad \Longrightarrow \quad$ Transpose of $\mathbf{A}$: $\quad \mathbf{A}^T = [\, a_{ji} \,]_{n \times m}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}^T_{3 \times 2} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2 \times 3}$$

Properties

- $(\mathbf{A}^T)^T = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
- $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$
- $(\alpha\mathbf{A})^T = \alpha\mathbf{A}^T, \quad \alpha$ is a scalar

Determinant

The determinant of a square matrix $\mathbf{A}$ is denoted as

$$|\mathbf{A}| = \det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Evaluation of Determinants

- $$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{21}a_{12}$$

  2×2 determinant

- $$|\mathbf{A}| = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

  To evaluate the 3×3 determinant, copy first two columns at right.

$$= +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Cramer's Rule

For the following system of n linear algebraic equations

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n &= b_2 \\ \cdots\ \ \cdots& \\ a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}\,x_n &= b_n \end{aligned}$$

the solutions are given by

$$x_i = \frac{\Delta_i}{\Delta} \qquad i = 1,\ 2,\ \ldots,\ n, \qquad \Delta \neq 0$$

where Δ is the determinant of coefficient matrix, Δ_i is the determinant of the coefficient matrix with the ith column replaced by the right-hand side vector, i.e.,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \qquad \Delta_i = \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & \underbrace{b_n}_{i\text{th column}} & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

Example

Solve the following system of linear algebraic equations

$$3x_1 + 4x_2 = 10$$

$$2x_1 - 5x_2 = -1$$

$$\Delta = \begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix} = 3 \cdot (-5) - 2 \cdot 4 = -23$$

$$\Delta_1 = \begin{vmatrix} 10 & 4 \\ -1 & -5 \end{vmatrix} = 10 \cdot (-5) - (-1) \cdot 4 = -46$$

$$\Delta_2 = \begin{vmatrix} 3 & 10 \\ 2 & -1 \end{vmatrix} = 3 \cdot (-1) - 2 \cdot 10 = -23$$

Apply Cramer's Rule

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{-46}{-23} = 2, \qquad x_2 = \frac{\Delta_2}{\Delta} = \frac{-23}{-23} = 1$$

Example

Solve the following system of linear algebraic equations

$$4y - 3z = 3$$

$$-x + 7y - 5z = 4$$

$$-x + 8y - 6z = 5$$

Determinant of the coefficient matrix

$$\Delta = \left|\begin{array}{ccc} 0 & 4 & -3 \\ -1 & 7 & -5 \\ -1 & 8 & -6 \end{array}\right| \begin{array}{cc} 0 & 4 \\ -1 & 7 \\ -1 & 8 \end{array}$$

$$= 0 \cdot 7 \cdot (-6) + 4 \cdot (-5) \cdot (-1) + (-3) \cdot (-1) \cdot 8$$

$$- (-1) \cdot 7 \cdot (-3) - 8 \cdot (-5) \cdot 0 - (-6) \cdot (-1) \cdot 4$$

$$= 0 + 20 + 24 - 21 - 0 - 24 = -1$$

$$\Delta_1 = \begin{vmatrix} 3 & 4 & -3 \\ 4 & 7 & -5 \\ 5 & 8 & -6 \end{vmatrix} \begin{matrix} 3 & 4 \\ 4 & 7 \\ 5 & 8 \end{matrix}$$

✍ Replace the first column by RHS vector.

$$= 3\cdot 7\cdot(-6) + 4\cdot(-5)\cdot 5 + (-3)\cdot 4\cdot 8 - 5\cdot 7\cdot(-3) - 8\cdot(-5)\cdot 3 - (-6)\cdot 4\cdot 4$$

$$= -126 - 100 - 96 + 105 + 120 + 96 = -1$$

$$\Delta_2 = \begin{vmatrix} 0 & 3 & -3 \\ -1 & 4 & -5 \\ -1 & 5 & -6 \end{vmatrix} \begin{matrix} 0 & 3 \\ -1 & 4 \\ -1 & 5 \end{matrix}$$

✍ Replace the second column by RHS vector.

$$= 0\cdot 4\cdot(-6) + 3\cdot(-5)\cdot(-1) + (-3)\cdot(-1)\cdot 5$$

$$- (-1)\cdot 4\cdot(-3) - 5\cdot(-5)\cdot 0 - (-6)\cdot(-1)\cdot 3$$

$$= 0 + 15 + 15 - 12 - 0 - 18 = 0$$

$$\Delta_3 = \begin{vmatrix} 0 & 4 & 3 \\ -1 & 7 & 4 \\ -1 & 8 & 5 \end{vmatrix} \begin{matrix} 0 & 4 \\ -1 & 7 \\ -1 & 8 \end{matrix}$$

✍ Replace the third column by RHS vector.

$$= 0\cdot7\cdot5 + 4\cdot4\cdot(-1) + 3\cdot(-1)\cdot8 - (-1)\cdot7\cdot3 - 8\cdot4\cdot0 - 5\cdot(-1)\cdot4$$

$$= 0 - 16 - 24 + 21 - 0 + 20 = 1$$

Apply Cramer's Rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-1}{-1} = 1, \qquad y = \frac{\Delta_2}{\Delta} = \frac{0}{-1} = 0, \qquad z = \frac{\Delta_3}{\Delta} = \frac{1}{-1} = -1$$

Systems of Homogeneous Linear Equations

When the right-hand side constants $b_1 = b_2 = \cdots = b_n = 0$, the system of linear algebraic equations is *homogeneous*

$$
\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n &= 0\\
a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n &= 0\\
\cdots\ \ \cdots&\\
a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}\,x_n &= 0
\end{aligned}
$$

- The system of homogeneous linear equations has zero solution, i.e.

$$x_1 = x_2 = \cdots = x_n = 0$$

- If the determinant of the coefficient matrix $\Delta \neq 0$, then the system of homogeneous linear equations does not have non-zero solutions.
- For the system of homogeneous linear equations to have non-zero solutions, the determinant of the coefficient matrix $\Delta = 0$.

Eigenvalues and Eigenvectors

Consider the following system of homogeneous linear equations

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x} \qquad \text{or} \qquad (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{*}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix, i.e.,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- For the system of homogeneous linear equations (*) to have non-zero solutions, the determinant of the coefficient matrix must be zero, i.e.,

  $$\left|\mathbf{A} - \lambda\,\mathbf{I}\right| = 0$$

  which is called the **characteristic equation**.

- The solutions (roots) λ of the characteristic equation are called **eigenvalues**.

- The corresponding solutions of system (*) are called **eigenvectors**.

Example

Find the value of λ such that the equations

$$\begin{aligned}(3-\lambda)\,x_1 - 3\,x_2 + x_3 &= 0\\ 2\,x_1 - (2+\lambda)\,x_2 + 2\,x_3 &= 0\\ -x_1 + 2\,x_2 - \lambda\,x_3 &= 0\end{aligned}$$

have non-zero solutions.

Characteristic equation (setting the determinant of the coefficient matrix to zero)

$$\Delta = \left|\begin{array}{ccc} 3-\lambda & -3 & 1 \\ 2 & -2-\lambda & 2 \\ -1 & 2 & -\lambda \end{array}\right| \begin{array}{cc} 3-\lambda & -3 \\ 2 & -2-\lambda \\ -1 & 2 \end{array}$$

$$\begin{aligned}&= (3-\lambda)\cdot(-2-\lambda)\cdot(-\lambda) + (-3)\cdot 2\cdot(-1) + 1\cdot 2\cdot 2\\ &\quad - (-1)\cdot(-2-\lambda)\cdot 1 - 2\cdot 2\cdot(3-\lambda) - (-\lambda)\cdot 2\cdot(-3)\\ &= -\lambda^3 + \lambda^2 - 2 = -\lambda^3 - \lambda^2 + 2\lambda^2 - 2\\ &= -\lambda^2(\lambda+1) + 2(\lambda-1)(\lambda+1) = -(\lambda+1)(\lambda^2-2\lambda+2) = 0\end{aligned}$$

$$\therefore \quad (\lambda+1)(\lambda^2-2\lambda+2)=0$$

- $\lambda+1=0 \implies \lambda=-1$

- $\lambda^2-2\lambda+2=0$:

$$\lambda=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot2}}{2\cdot1}=\frac{2\pm\sqrt{-4}}{2}=1\pm\mathrm{i}, \quad \mathrm{i}=\sqrt{-1}$$

Alternative Method for Solving the Characteristic Equation $-\lambda^3+\lambda^2-2=0$

By trial-and-error, find a root of the characteristic equation

$$-(-1)^3+(-1)^2-2=0 \implies \lambda=-1 \text{ is a root} \implies (\lambda+1) \text{ is a factor.}$$

Use long division to find the other factor as follows

$$
\begin{array}{r|rrrrrrrl}
 & -\lambda^2 & + & 2\lambda & - & 2 & & & \\
\hline
\lambda+1 & -\lambda^3 & + & \lambda^2 & & & - & 2 & \\
 & -\lambda^3 & - & \lambda^2 & & & & & (- \\
\cline{2-9}
 & & & 2\lambda^2 & & & - & 2 & \\
 & & & 2\lambda^2 & + & 2\lambda & & & (- \\
\cline{4-9}
 & & & & - & 2\lambda & - & 2 & \\
 & & & & - & 2\lambda & - & 2 & (- \\
\cline{6-9}
 & & & & & & & 0 &
\end{array}
$$

The characteristic equation becomes $(\lambda+1)(-\lambda^2+2\lambda-2)=0.$

Example

Find the eigenvalues λ and the corresponding eigenvectors of

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\begin{bmatrix}1-\lambda & 4\\ 1 & -2-\lambda\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

The characteristic equation is

$$\begin{vmatrix}1-\lambda & 4\\ 1 & -2-\lambda\end{vmatrix}=(1-\lambda)(-2-\lambda)-1\cdot 4=\lambda^2+\lambda-6=(\lambda+3)(\lambda-2)=0$$

The two eigenvalues are $\lambda_1=-3,\quad \lambda_2=2.$

- $\lambda=\lambda_1=-3$:

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{v}_1=\begin{bmatrix}4 & 4\\ 1 & 1\end{bmatrix}\begin{Bmatrix}v_{11}\\ v_{21}\end{Bmatrix}=\mathbf{0}\quad\Longrightarrow\quad v_{11}+v_{21}=0$$

Taking $v_{21}=-1$, then $v_{11}=-v_{21}=1 \quad\Longrightarrow\quad \mathbf{v}_1=\begin{Bmatrix}v_{11}\\ v_{21}\end{Bmatrix}=\begin{Bmatrix}1\\ -1\end{Bmatrix}$

✍ There is one equation for two unknowns.

One unknown can be solved in terms of the other unknown.

- $\lambda = \lambda_2 = 2$:

$$(\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{v}_2 = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix} = \mathbf{0} \quad \Longrightarrow \quad v_{12} - 4v_{22} = 0$$

Taking $v_{22} = 1$, then $v_{12} = 4v_{22} = 4 \quad \Longrightarrow \quad \mathbf{v}_2 = \begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix} = \begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$